FLOWS OF THE HELE–SHAW TYPE BETWEEN COAXIAL SHELLS OF REVOLUTION

P. A. Novikov, L. Ya. Lyubin,
and V. I. Novikova

UDC 532.54

Flows are considered in narrow gaps, whose mean surfaces are being formed by two planar parts of a sphere, a right circular cone, an ellipsoid, one- and two-sheet ellipsoids, a paraboloid, and a torus.

The application of the general results obtained in [1] can be illustrated on flows in channels with gaps. The mean surface in such channels is the surface of revolution $\xi$ = const, on which can be introduced meridional ($\xi^1 = \eta$) and angular ($\xi^2 = \varphi$) coordinates, while the dimensionless Lamé coefficients are independent of the angular coordinate, i.e., $H_1 = H_\eta(\eta, \xi)$; $H_2 = H_\varphi(\eta, \xi)$. In this case Eq. (14) of [1] is written in the form

$$\frac{H_\varphi}{H_\eta}\frac{\partial}{\partial\eta}\left(\frac{H_\varphi}{H_\eta}\frac{\partial\Pi^{(0)}}{\partial\eta}\right)+\frac{\partial^2\Pi^{(0)}}{\partial\varphi^2}=0, \tag{1}$$

and here one naturally introduces the new independent variable

$$\chi_0(\eta,\ \xi)=\int\limits_{\eta_0}^{\eta}\frac{H_\eta(\eta,\ \xi)}{H_\varphi(\eta,\ \xi)}\,d\eta\quad(\xi=\mathrm{const}), \tag{2}$$

which makes it possible to replace Eq. (1) by the two-dimensional Laplace equation

$$\frac{\partial^2\Pi^{(0)}}{\partial\chi_0^2}+\frac{\partial^2\Pi^{(0)}}{\partial\varphi^2}=0. \tag{3}$$

If, in particular, a narrow gap channel is considered with two sides restricted by the planar segments $\eta = \eta_1$ and $\eta = \eta_2$, on which are given boundary conditions of the first $[\Pi(\eta_i,\varphi)=\Pi_i(\varphi)]$, second $[\partial\Pi(\eta_i,\varphi)/\partial\eta = f_i(\varphi)]$, or third $[\partial\Pi(\eta_i,\varphi)/\partial\eta - \varkappa_i\Pi(\eta_i,\varphi) = f_i(\varphi)]$ kind, then the solution of the corresponding problem is easily obtained by means of the method of Fourier variable separation, and

$$\Pi^{(0)}=C_0\chi_0+\sum_{n=1}^{\infty}[C_n\exp(n\chi_0)+C_{-n}\exp(-n\chi_0)]\cos(n\varphi). \tag{4}$$

The constants $C_0$, $C_n$, and $C_{-n}$ are found, as usual, from the solutions of the linear algebraic equations, which are obtained by substituting expression (4) into the boundary conditions. In view of (2) and Eqs. (13) of [1] the total fluid discharge through the annular channel considered it

$$Q=-\frac{2}{3}r_0Vh\int\limits_0^{2\pi}\frac{H_\varphi}{H_\eta}\frac{\partial\Pi^{(0)}}{\partial\eta}\,d\varphi=-\frac{2}{3}r_0VhC_0\int\limits_0^{2\pi}\frac{\partial\chi_0}{\partial\eta}\frac{H_\varphi}{H_\eta}\,d\varphi, \tag{5}$$

i.e.,

$$Q=-\frac{4\pi}{3}r_0VhC_0\quad\text{or}\quad C_0=-\frac{3Q}{4\pi r_0Vh}.$$

Thus, for Re = 0 the pressure drop averaged over the angular coordinate $\varphi$ at the portion of the narrow gap channel between cross sections $\eta = \eta_1$ and $\eta = \eta_2$

$$\Delta p^{(0)}=\frac{r_0\rho\nu V}{h^2}[\Pi^{(0)}(\eta_1)-\Pi^{(0)}(\eta_2)]=\frac{3\rho\nu Q}{4\pi h^3}[\chi_0(\eta_2,\ \xi)-\chi_0(\eta_1,\ \xi)]. \tag{6}$$

In the absence of inhomogeneities in the angular coordinate $\varphi$ in the boundary conditions, the coefficients $C_n$, $C_{-n} = 0$, and expression [6] provides not only the averaged, but also the

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 6, pp. 940–948, June, 1987. Original article submitted March 6, 1986.

local values of the pressure drop for arbitrary meridional cross sections $\varphi$ = const.

If at the cutoffs of the narrow gap channel $\eta = \eta_1$ and $\eta = \eta_2$ are given the specific fluid fluxes $2q_\eta = 2q_{\xi_1}\left(= -\frac{1}{3}\frac{1}{H_\eta}\frac{\partial\Pi}{\partial\eta}\right.$ boundary conditions of the second kind), then by (20) of [1]

$$\Pi^{(1)} = -\frac{3}{35}\left[\left(\frac{1}{H_\eta}\frac{\partial\Pi^{(0)}}{\partial\eta}\right)^2 + \left(\frac{1}{H_\varphi}\frac{\partial\Pi^{(0)}}{\partial\varphi}\right)^2\right]$$

and the pressure component quadratic in the discharge $p^{(1)} = \rho V^2\Pi^{(1)}$ is determined by the expression

$$p^{(1)} = -\frac{27}{560}\frac{\rho Q^2}{\pi^2 r_0^2 h^2}\chi_1 - \frac{3}{35}\frac{\rho V^2}{H_\varphi^2}\sum_{n=1}^{\infty}\{2nC_0[C_n\exp(n\chi_0) +$$

$$+ C_{-n}\exp(-n\chi_0)]\cos(n\varphi) + n^2[C_n^2\exp(2n\chi_0) + C_{-n}^2\exp(-2n\chi_0)] - 2n^2C_nC_{-n}\cos(2n\varphi)\};\quad \chi_1 = H_\varphi^{-2}. \tag{7}$$

To calculate the cubic pressure component $p^{(2)}$ in the fluid discharge for boundary conditions of the second kind one may use Eq. (26) of [1], which in the case under consideration is written in the form

$$\Pi^{(2)} = -3\Phi^{(2)} + \frac{104\,C_0^3}{363\,825}\left(1 + \frac{\chi}{4}\right)\frac{1}{H_\varphi^4 H_\eta}\frac{\partial H_\varphi}{\partial\eta}.$$

The potential $\Phi^{(2)}$ satisfies the equation

$$\frac{\partial}{\partial\eta}\left(\frac{H_\varphi}{H_\eta}\frac{\partial\Phi^{(2)}}{\partial\eta}\right) = 0$$

and the boundary conditions equivalent to (25) of [1]

$$\frac{H_\varphi}{H_\eta}\frac{\partial\Phi^{(2)}}{\partial\eta} = -\frac{26\,C_0^3}{363\,825}\left\{\frac{1}{2}\frac{H_\varphi}{H_\eta}\frac{\partial}{\partial\eta}\left[\frac{\chi_0}{H_\varphi H_\eta}\frac{\partial}{\partial\eta}\right.\right. \times$$

$$\left.\left.\times\left(\frac{H_\varphi}{H_\eta}\frac{\partial}{\partial\eta}\frac{1}{H_\varphi^2}\right)\right] - \frac{1}{H_\varphi H_\eta}\frac{\partial}{\partial\eta}\left(\frac{H_\varphi}{H_\eta}\frac{\partial}{\partial\eta}\frac{1}{H_\varphi^2}\right)\right\}$$

at the cross section $\eta = \eta_1$ and $\eta = \eta_2$. However, the right hand side of the latter relation vanishes identically. Consequently, $\Phi^{(2)}$ = const. It can be assumed that $\Phi^{(2)} = 0$. Thus, the pressure component $p^{(2)} = r_0\rho\nu V h^{-2}Re^2\Pi^{(2)}$ is determined by the expression

$$p^{(2)} = -\frac{13}{431\,200}\frac{\rho Q^3}{\pi^3 r_0^4 h\nu}\chi_2;\quad \chi_2 = \frac{4+\chi_0}{H_\varphi^4 H_\eta}\frac{\partial H_\varphi}{\partial\eta} \tag{8}$$

and, confining ourselves to the expansion terms written down, then for a given fluid discharge Q

$$\Delta p = \sum_{k=0}^{2}[p^{(k)}(\eta_1) - p^{(k)}(\eta_2)]. \tag{9}$$

We illustrate the application of the relations obtained. The simplest example is a narrow gap with a mean surface obtained from a sphere after removing two segments, cutoff by parallel planes (a spherical belt). For convenience of comparison with flows in narrow gap channels, whose mean surfaces are ellipsoids of revolution, we use a spherical coordinate system $\xi$, $\eta$, $\varphi$ (Fig. 1a) of not exactly traditional shape. The surfaces $\xi$ = const are spheres of radius $r_0\xi$; $\xi = 1$ corresponds to the mean surface of the narrow gap channels. The surfaces $\eta$ = const are cones, whose generating lines form an angle $\theta = \arccos\eta$ with the axis $z$ $(-1 \leqslant \eta \leqslant 1)$. The surfaces $\varphi$ = const are planes, forming an angle $\varphi$ with the plane $y = 0$ $(0 \leqslant\varphi\leqslant 2\pi)$. Here $R = \sqrt{x^2 + y^2} = r_0\xi\sqrt{1-\eta^2}$, $z = r_0\xi\eta$, $H_\eta = \xi(1-\eta^2)^{-1/2}$, $H_\varphi = \xi\sqrt{1-\eta^2}$. Thus, the pressure coefficients are

$$\chi_0 = \frac{1}{2}\ln\frac{1+\eta}{1-\eta};\quad \chi_1 = \frac{1}{1-\eta^2};\chi_2 = \frac{\eta}{(1-\eta^2)^2}\left(4 + \frac{1}{2}\ln\frac{1+\eta}{1-\eta}\right). \tag{10}$$

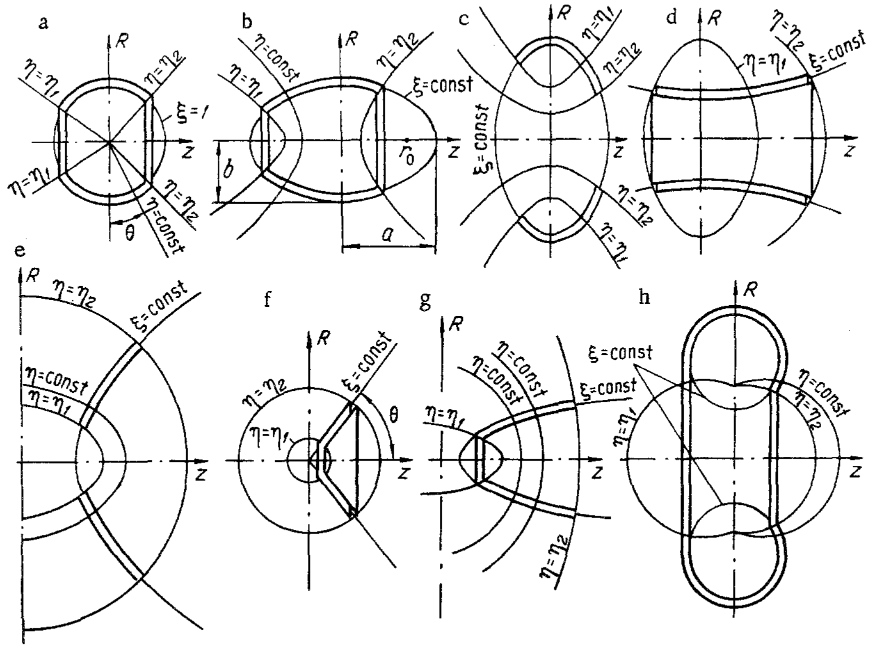


Fig. 1. Schemes of nonplanar slot gaps, formed by coaxial shells of revolution.

It is now natural to consider annular channels, whose mean surface is part of a spheroid (an ellipsoid of revolution), formed by two planes perpendicular to the symmetry axis. Denoting by the letter $k = (a/b)$ the ratio of lengths of the spheriod axis, consider separately the two cases: $k > 1$ and $k < 1$. In the first case we use the prolate spheriodal coordinates $\xi$, $\eta$, $\varphi$ [2], which are formed by the rotation of ellipsoidal coordinates around the major axis of the ellipse. If the foci of the spheroid are located at the points $x = y = R = 0$, $z = \pm r_0$, then the corresponding system is determined as follows (Fig. 1b):

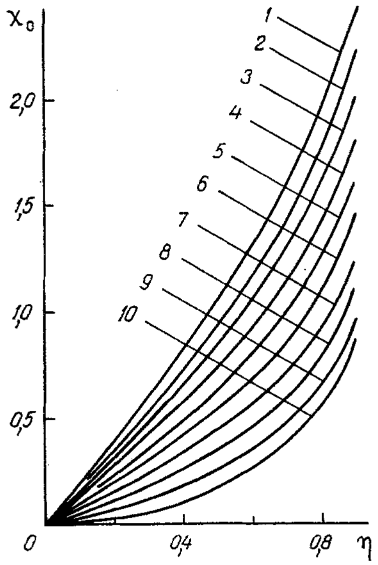
$$R = r_0\sqrt{(\xi^2-1)(1-\eta^2)};\quad z = r_0\xi\eta;\quad a = r_0\xi;\quad b = r_0\sqrt{\xi^2-1};$$
$$k = \frac{\xi}{\sqrt{\xi^2-1}};\quad H_\eta = \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}};\quad H_\varphi = \sqrt{(\xi^2-1)(1-\eta^2)}. \tag{11}$$

The surfaces $\xi$ = const are here prolate spheroids $(1 < \xi < \infty)$. The surface $\eta$ = const is a two-sheet hyperboloid of revolution with foci at the points $z = \pm r_0$, forming an asymptotic cone forming with the z axis an angle $\theta = \arccos \eta$ $(-1 \leqslant \eta \leqslant 1)$. By (2), (7), and (8)

$$\chi_0 = \frac{1}{2}\ln\left(\frac{\sqrt{\xi^2-1}+\sqrt{\xi^2-\eta^2}}{1-\eta} + \frac{1}{\sqrt{\xi^2-1}}\right) -$$
$$\frac{1}{2}\ln\left(\frac{\sqrt{\xi^2-1}+\sqrt{\xi^2-\eta^2}}{1+\eta} + \frac{1}{\sqrt{\xi^2-1}}\right) + \frac{1}{\sqrt{\xi^2-1}}\arcsin\frac{\eta}{\xi};$$
$$\chi_1 = \frac{1}{(\xi^2-1)(1-\eta^2)};\quad \chi_2 = \frac{\eta\,[4+\chi_0(\eta)]}{(1-\eta^2)^2\sqrt{(\xi^2-1)^3(\xi^2-\eta^2)}}. \tag{12}$$

For $k < 1$ one can use oblate spheroidal coordinates $\xi$, $\eta$, $\varphi$ [2], which are obtained by rotating the cofocal elliptic coordinates around the minor axis. The corresponding system is determined as (Fig. 1c):

$$R = r_0\sqrt{(\xi^2+1)(1-\eta^2)};\quad z = r_0\xi\eta;\quad a = r_0\xi;\quad b = r_0\sqrt{\xi^2+1};$$
$$k = \frac{\xi}{\sqrt{\xi^2+1}};\quad H_\eta = \sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}};\quad H_\varphi = \sqrt{(\xi^2+1)(1-\eta^2)}. \tag{13}$$



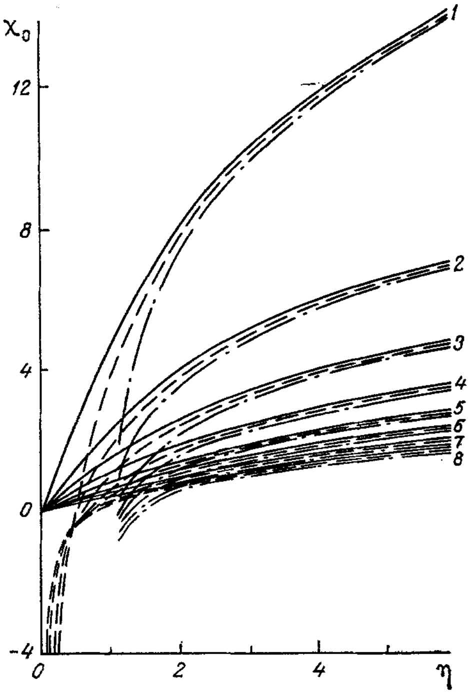




Fig. 2 Fig. 3

Fig. 2. Function $\chi_0$, characterizing the pressure in a narrow gap channel, vs the coordinate $\eta$, proportional to the distance between the cutoff and the equatorial cross section $\eta = 0$ ($z = 0$), for channels with mean surface of ellipsoidal (scheme b, Fig. 1, curves 1-5; scheme c, curves 7-10; and spherical shape (scheme a, curve 6) with $k = a/b$: 1) 2, 2) 1.8; 3) 1.6; 4) 1.4; 5) 1.2; 6) 1; 7) 0.8; 8) 0.6; 9) 0.4; 10) 0.2.

Fig. 3. Dependence of $\chi_0$ on $\eta$ for na-row gap channels with a mean surface of hyperbolic (solid lines, scheme d, Fig. 1; dashed-dotted, scheme e) and cone shapes (dotted lines, scheme f) for angles $\theta$: 1) 10°; 2) 20; 3) 30; 4) 40; 5) 50; 6) 60; 7) 70; 8) 80°.

The surface $\xi =$ const is an oblate spheroid, the length of whose axis of rotation is $2r_0\xi$, and the radius in the equatorial plane being $r_0\sqrt{\xi^2+1}$ $(0 < \xi < \infty)$. The surface $\eta =$ const is a one-sheet hyperboloid of revolution, whose axis coincides with the z axis, and forming an asymptotic cone inclined by an angle $\theta = \arccos \eta$ to this axis $(-1 < \eta < 1)$. In the case given

$$\chi_0 = \frac{1}{2}\ln\left(\frac{\sqrt{\xi^2+1}+\sqrt{\xi^2+\eta^2}}{1-\eta} - \frac{1}{\sqrt{\xi^2+1}}\right) - \frac{1}{2}\ln\left(\frac{\sqrt{\xi^2+1}+\sqrt{\xi^2+\eta^2}}{1+\eta} - \frac{1}{\sqrt{\xi^2+1}}\right) - \frac{1}{\sqrt{\xi^2+1}}\ln\frac{\sqrt{\xi^2+\eta^2}+\eta}{\xi};$$

$$\chi_1 = \frac{1}{(\xi^2+1)(1-\eta^2)};\quad \chi_2 = \frac{\eta\,[4+\chi_0(\eta)]}{(1-\eta^2)^2\sqrt{(\xi^2+1)^3(\xi^2+\eta^2)}}. \tag{14}$$

Figure 2 shows results of calculating the viscous component of the resistance $p^{\circ}$ (the function $\chi_0$) for various k values.

The same coordinate systems can also be used to analyze flows in narrow gap channels, whose mean surfaces are other coordinate surfaces. If, for example, the mean surface corresponds to a formation by two planes perpendicular to the symmetry axis as part of a one-sheet hyperboloid of revolution, then one can use oblate spheroidal coordinates (13), placed in the locations $\xi$ and $\eta$, so that (Fig. 1d)

$$R = r_0\sqrt{(1-\xi^2)(\eta^2+1)};\quad z = r_0\xi\eta;\quad H_\varphi = \sqrt{(1-\xi^2)(\eta^2+1)};$$
$$H_\eta = \sqrt{\frac{\xi^2+\eta^2}{\eta^2+1}}.$$

The surface $\xi$ = const is here a one-sheet hyperboloid of revolution, forming an asymptotic cone inclined by the angle $\theta = \arccos\xi$ to the z rotation axis ($|\xi| \leqslant 1$). The surface $\eta$ = const is an oblate spheroid, the length of whose axis of rotation equals $2r_0\eta$, while the radius in the equatorial plane equals $r_0\sqrt{\eta^2+1}$ ($0 < \eta < \infty$). Using (2), (7), and (8), we obtain

$$\chi_0 = \frac{1}{2}\ln\frac{\sqrt{\xi^2+\eta^2}-\eta\sqrt{1-\xi^2}}{\sqrt{\xi^2+\eta^2}+\eta\sqrt{1-\xi^2}} + \frac{1}{\sqrt{1-\xi^2}}\ln\frac{\eta+\sqrt{\xi^2+\eta^2}}{\xi};$$
$$\chi_1 = \frac{1}{(1-\xi^2)(\eta^2+1)};\quad \chi_2 = \frac{\eta\,(4+\chi_0(\eta))}{(\eta^2+1)^2\sqrt{(1-\xi^2)^3(\xi^2+\eta^2)}}. \tag{15}$$

If the mean surface is cut by two planes, perpendicular to the axis of symmetry, part of the two-sheet hyperboloid of rotation, it is natural to use the prolate spheroidal coordinates (11), putting in them according to the notation used here the coordinates $\xi$ and $\eta$, so that now

$$R = r_0\sqrt{(\eta^2-1)(1-\xi^2)};\quad z = r_0\xi\eta \text{ (Fig. 1d)};$$
$$H_\eta = \sqrt{\frac{\eta^2-\xi^2}{\eta^2-1}};\qquad H_\varphi = \sqrt{(1-\xi^2)(\eta^2-1)}.$$

Here the surface $\xi$ = const is a two-sheet hyperboloid of revolution, forming an asymptotic cone inclined by an angle $\theta = \arccos\xi$ to the z axis ($|\xi| \leqslant 1$), while the surfaces $\eta$ = const are prolate spheroids with focal distance $2r_0$ ($1 \leqslant \eta \leqslant \infty$). In the case given

$$\chi_0 = \frac{1}{2}\left[\ln\left(\frac{1}{\sqrt{1-\xi^2}} - \frac{\sqrt{\eta^2-\xi^2}+\sqrt{1-\xi^2}}{\eta+1}\right) - \right.$$
$$\left. -\ln\left(\frac{1}{\sqrt{1-\xi^2}} + \frac{\sqrt{\eta^2-\xi^2}+\sqrt{1-\xi^2}}{\eta-1}\right) - \ln\frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}}\right] + \frac{1}{\sqrt{1-\xi^2}}\ln\frac{\sqrt{\eta^2-\xi^2}+\eta}{\xi};$$
$$\chi_1 = \frac{1}{(1-\xi^2)(\eta^2-1)};\quad \chi_2 = \frac{\eta\,[4+\chi_0(\eta)]}{(\eta^2-1)\sqrt{(1-\xi^2)^3(\eta^2-\xi^2)}}. \tag{16}$$

If the mean surface of the annular channel is cut by two planes perpendicular to the axis part of the right circular cone, then one can use spherical coordinates, also choosing the location of coordinates $\xi$ and $\eta$, so that now (Fig. 1f) $R = r_0\eta\sqrt{1-\xi^2}$, $z = r_0\xi\eta$, $H_\eta = 1$, $H_\varphi = \eta\sqrt{1-\xi^2}$. The surface $\xi$ = const is a cone, inclined by an angle $\theta = \arccos\xi$ to the z axis ($-1 \leqslant \xi \leqslant 1$), while the surfaces $\eta$ = const are spheres of radius $r_0\eta$, whose centers coincide with apex of the cone ($0 < \eta < \infty$). Now

$$\chi_0 = \frac{\ln\eta}{\sqrt{1-\xi^2}};\quad \chi_1 = \frac{1}{(1-\xi^2)\,\eta^2};\quad \chi_2 = \frac{4+\chi_0(\eta)}{\sqrt{(1-\xi^2)^3}}. \tag{17}$$

The dependence $\chi_0(\eta)$ for hyperboloids of one-sheet (solid lines), two-sheet (dotted-dashed), and right circular cones (dotted) for various values of the angle $\Theta$ is shown in Fig. 3. It is seen from the plots and the corresponding equations that the flows in the three groups of annular channels considering equations that the flows in the three groups of annular channels considered above have common asymptotes for $\eta \to \infty$.

To analyze arbitrary flows in a conic narrow gap it is advisable to expand the mean surface on a plane, and use in solving the corresponding boundary value problems methods of the theory of functions of a complex variable. An example of this approach is available in [3].

If the mean surface of the annular channel is part of a paraboloid of revolution, cut by two planes perpendicular to the symmetry axis z, one can use a parabolic coordinate sys-

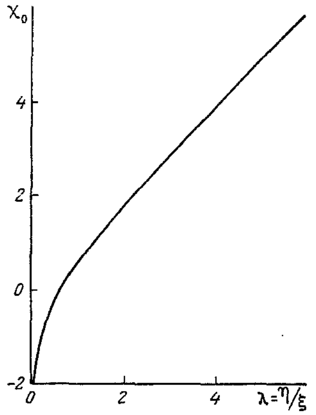



Fig. 4. Dependence of $\chi_0$ on $\eta/\xi$ for narrow gap channels with a mean surface of parabolic shape (scheme g, Fig. 1).

tem $\xi$, $\eta$, $\varphi$ [2], formed by rotating two-dimensional parabolic coordinates about their axis. This system is determined as follows (Fig. 1g): $R=r_0\xi\eta$; $z=\frac{r_0}{2}(\eta^2-\xi^2)$; $H_\eta=\sqrt{\xi^2+\eta^2}$; $H_\varphi=\xi\eta$. The surfaces $\xi$ = const and $\eta$ = const of paraboloids of revolution ($0 < \xi$, $\eta < \infty$). By (2), (7), and (8)

$$\chi_0=\sqrt{1+\frac{\eta^2}{\xi^2}}-\ln\left(\frac{\xi}{\eta}+\sqrt{1+\frac{\xi^2}{\eta^2}}\right);$$
$$\chi_1=\frac{1}{\xi^2\eta^2};\quad \chi_2=\frac{4+\chi_0(\eta)}{\xi^3\eta^4\sqrt{\xi^2+\eta^2}}. \tag{18}$$

The dependence of the coefficient $\chi_0$ on the coordinate $\eta/\xi$ is shown in Fig. 4. For $\eta/\xi > 3$ it is almost linear, since the first equation corresponds to the asymptotic representation

$$\chi_0=\frac{\eta}{\xi}\left[1-\frac{1}{2}\left(\frac{\xi}{\eta}\right)^2+O\left(\frac{\xi^3}{\eta^3}\right)\right],\quad \frac{\xi}{\eta}\to 0.$$

If the mean surface of the annular channel is part of a torus, cut by two planes perpendicular to the symmetry axis z, one can use the toroidal coordinate system $\xi$, $\eta$, $\varphi$ [2], formed by rotating bipolar coordinates around the perpendicular passing through the middle of the segment combining the poles. This system is determined as follows (Fig. 1g):

$$R=\frac{r_0\,\mathrm{sh}\,\xi}{\mathrm{ch}\,\xi-\cos\eta};\quad z=\frac{r_0\sin\eta}{\mathrm{ch}\,\xi-\cos\eta};$$
$$H_\eta=\frac{1}{\mathrm{ch}\,\xi-\cos\eta};\quad H_\varphi=\frac{\mathrm{sh}\,\xi}{\mathrm{ch}\,\xi-\cos\eta}.$$

The surface $\xi$ = const is a torus, whose circular radius lies in the x, y plane, has a center at the origin of coordinates and a radius $r_0\mathrm{cth}\,\xi$, while the circular transverse cross section has a radius $r_0/\mathrm{sh}\,\xi$ ($0 < \xi < \infty$). The surface $\eta$ = const is for $\eta < \pi$ part of a sphere of radius $r_0\mathrm{cosec}\,\eta$ with center at the point $x = y = 0$, $z = r_0\mathrm{ctg}\,\eta$, located over the x, y plane ($z > 0$). The remaining part of the same sphere corresponds to the surface $\eta' = 2\pi - \eta$ ($0 \leqslant \eta \leqslant 2\pi$). Separating their line is a neighborhood of radius $r_0$ with center at the origin of coordinates. It corresponds to $\xi = \infty$. The other limit $\xi = 0$ corresponds to the z axis. Part of the x, y plane, lying within the neighborhood $\xi = \infty$, corresponds to the surface $\eta = \pi$, and the remaining part of this plane — to the surface $\eta = 0$ or $\eta = 2\pi$. Using (2), (7), and (8), we obtain

$$\chi_0=\frac{\eta}{\mathrm{sh}\,\xi};\quad \chi_1=\frac{(\mathrm{ch}\,\xi-\cos\eta)^2}{\mathrm{sh}^2\,\xi};\quad \chi_2=\frac{(4\,\mathrm{sh}\,\xi+\eta)(\mathrm{ch}\,\xi-\cos\eta)^3}{4\,\mathrm{sh}^4\,\xi}.$$

The asymptote constructed can be used for engineering calculations of the flow of a viscous fluid in a narrow gap channel between nonplanar surfaces with Re << 1. The estimate performed has shown that for engineering calculations we can restrict ourselves to the first two expansion terms. Thus, in the absence of inhomogeneities in the angular coordinate $\varphi$ in the boundary conditions the first iteration is $p^{(1)}=(13\cdot 10^{-2}\mathrm{Re}\frac{\chi_1}{\chi_0})p^{(0)}$ and the second is $p^{(2)}=(16\cdot 10^{-5}$

$\mathrm{Re}^2 \frac{\chi_2}{\chi_0}) p^{(0)}$ . For example, at Re = 0.1, in all cases considered in the present study $p^{(1)}$ is no more than 12% of $p^{(0)}$, while $p^{(2)}$ does not exceed 0.1% of $p^{(0)}$, i.e., in neglecting the third expansion term the error in the calculation does not exceed 0.1%.

## NOTATION

$R, z, \varphi$, cylindrical coordinates of points of the mean surface of an annular channel; $\xi, \eta, \varphi$, dimensionless orthogonal coordinate system; $\xi$ = const, surface of revolution corresponding to the mean surface of the channel; $\eta, \varphi$, meridional and angular coordinates at the mean surface, $\eta = \eta_1$ and $\eta = \eta_2$, boundaries of the annular channel; $H_\eta(\eta, \xi)$, $H_\varphi(\eta, \xi)$, dimensionless Lamé coefficients; p, pressure; $\rho$, fluid density; $\nu$, kinetic viscosity coefficient, V, characteristic flow velocity; Q, total fluid discharge through the annular channel; $C_0$, $C_n$, $C_{-n}$, coefficients of expansion (4) in a Fourier series; $\chi_n(\eta, \xi)$, asymptotic expansion coefficients of the dimensionless pressure $\Pi$, defined by Eqs. (2), (7), and (8) for n = 0, 1, 2; and $\Delta p$, pressure drop in the channel.

## LITERATURE CITED

1. P. A. Novikov and L. Ya. Lyubin, Inzh.-Fiz. Zh., 52, No. 4, 569-575 (1987).
2. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York (1953).
3. P. A. Novikov, L. Ya. Lyubin, and É. K. Snezhko, Inzh.-Fiz. Zh., 29, No. 3, 469-478 (1975).

# STABILITY OF THE BOUNDARY LAYER OF LIQUID UNDER A NONUNIFORM TEMPERATURE DISTRIBUTION OF THE SURFACE

Yu. B. Lebedev and V. M. Fomichev

UDC 532.526

We study the effect of a longitudinal gradient of the surface temperature on the stability of the boundary layer of an incompressible liquid. A comparison shows a good agreement of the results with experimental data.

Only a relatively small number of works have been devoted to the problem of stability and the transition to the turbulent regime in a boundary layer of an incompressible liquid at a surface with heat-exchange [1-9]. It was noted in the first investigations (which were carried out for water [1, 2]) that the surface temperature affects considerably the transition process. The nature of this influence is opposite to that which is observed in gases. This is caused by the decrease of water viscosity with increasing temperature. Cooling causes the appearance of an inflection point in the mean velocity profile near the wall and, consequently, it destabilizes the flow. Heating, on the other hand, gives a fuller velocity profile and, accordingly, it stabilizes the flow.

A sufficiently strong dependence of the stability characteristics (the minimum critical Reynolds numbers and the coefficients of spatial growth of the perturbations) for a surface layer of water on the superheating of the surface gives grounds for expectations that, for an appropriate temperature distribution along the surface, a considerable increase or decrease of the flow stability can be obtained. Detailed investigations of this problem can play an important role in the solution of the control of the boundary layer. The practical importance of the problem is confirmed also by the results of the experimental work [8] which have a preliminary character and indicate that the transition of a laminar boundary layer to a turbulent one depends, to a considerable degree, on the longitudinal temperature gradient at the wall.

In the present work we study the effect of nonuniformity of the surface temperature distribution on the development of small perturbations in a laminar boundary layer of an incompressible liquid. It is shown that, when the total heat flux remains unchanged, the posi-

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 6, pp. 948-953, June, 1987. Original article submitted March 25, 1986.